

Theory of Neutrino Flavor Mixing

Christian Y. Cardall

*Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831-6354;
Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996-1200;
Joint Institute for Heavy Ion Research, Oak Ridge National Laboratory, Oak Ridge, TN
37831-6374*

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Abstract

The depth of our theoretical understanding of neutrino flavor mixing should match the importance of this phenomenon as a herald of long-awaited empirical challenges to the standard model of particle physics. After reviewing the familiar, simplified quantum mechanical model and its flaws, I sketch the deeper understanding of both vacuum and matter-enhanced flavor mixing that is found in the framework of scattering theory. While the simplified model gives the “correct answer” for atmospheric, solar, and accelerator/reactor neutrino phenomena, I argue that a key insight from the deeper picture will simplify the treatment of neutrino transport in astrophysical environments—supernovae, for example—in which neutrinos play a dynamically important role.

I. INTRODUCTION

Because the standard model of particle physics admits no neutrino mass, any evidence of flavor mixing of massive neutrinos challenges this reigning paradigm. While the extensions of the standard model that are required to give mass to the neutrino can be relatively trivial, patterns of neutrino mass and flavor mixing may provide interesting clues to the high-energy unification of the separate interactions that the standard model describes with such precision at currently accessible energies.

There are various phenomena that beg to be interpreted as neutrino flavor mixing. Some hints and constraints come from observations of solar neutrinos, reactor and accelerator neutrino experiments, and consideration of the role of neutrinos in supernovae and Big Bang nucleosynthesis [1]. Of particular note is the observation, in the Super-Kamiokande detector, of large numbers of neutrinos produced in the impacts of cosmic rays upon Earth’s atmosphere [2]. These observations establish, with high statistics, an asymmetry between the upward and downward ν_μ fluxes. This asymmetry is naturally explained as resulting from the sinusoidal flux variation with path length expected to result from neutrino flavor mixing. It may well be remembered in the future as one of the first important pieces of

empirical evidence of physics beyond the standard model. Long-baseline terrestrial neutrino experiments will test the flavor mixing interpretation of the Super-Kamiokande data; initial results from the first of these involving a ν_μ beam (the K2K experiment) show a ν_μ deficit [3], but more data are needed for a definitive confirmation.

(About three months after this conference, the SNO collaboration announced an empirical milestone regarding solar neutrinos. Comparison of the detected rate of neutrino absorption on deuterium (a charged current process) with the rate of elastic scattering of electrons (which has contributions from both charged and neutral currents) provides evidence that a portion of the solar neutrino flux is comprised of ν_μ or ν_τ [4].)

Our theoretical understanding of neutrino flavor mixing should be sufficiently deep and secure to be worthy of its important role as a phenomenological beacon, signaling towards us from the energetically distant shores of grand unification. To this end, I first review the familiar, simplified quantum mechanical model. This model embodies the simple heart of the physics, but also contains flaws that appear under close examination. Next I sketch the deeper understanding of both vacuum and matter-enhanced flavor mixing that is found in considering neutrino production, propagation, and detection as a single unified scattering process. In conclusion, I argue that while the quantitative differences predicted by the deeper framework are not discernable in current experiments, a fundamental insight from this framework will be helpful in conceptualizing neutrino transport in astrophysical phenomena—such as supernovae—in which the free-streaming approximation breaks down.

II. SIMPLIFIED QUANTUM MECHANICAL MODEL

In describing massless neutrinos, the standard model employs only one of two spin-1/2 representations of the Lorentz group. These “left-handed” neutrino fields annihilate and create neutrino states of negative helicity and antineutrino states of positive helicity. For example, a low-energy effective interaction responsible for the emission or absorption of neutrinos via interactions with nucleons can be written as a product of leptonic and hadronic currents:

$$\mathcal{L}_I = -g \bar{\alpha} \gamma^\mu (1 - \gamma_5) \nu_\alpha \bar{p} \gamma_\mu (1 - g_A \gamma_5) n + \text{H.c.}, \quad (1)$$

where g and g_A are coupling constants, and α , ν_α , p , and n are Dirac fields that respectively annihilate particles and create antiparticles of the following types: charged leptons of flavor α ; neutrinos associated with those charged leptons; protons; and neutrons. The $(1 - \gamma_5)$ factor projects the one needed spin-1/2 representation of the Lorentz group from the Dirac spinor ν_α . In a description of massless neutrinos, this so-called left-handed field $\nu_{\alpha L} = \frac{1}{2}(1 - \gamma_5)\nu_\alpha$ is all that appears in the free-field part of the Lagrangian as well.

While empirically undetectable when the standard model was put together, there is reason to believe that neutrinos have mass. In contrast to electrodynamics—whose observed gauge invariance requires a vanishing photon mass—there is no symmetry principle¹ to

¹Except, perhaps, lepton universality; see Eq. (10), and the surrounding discussion and footnote.

demand a vanishing neutrino mass. If one subscribes to the dictum that whatever is not forbidden is not just allowed, but *mandatory*, then neutrino mass is expected at some level.

A trivial extension to the standard model that allows neutrino mass is the introduction of right-handed neutrino fields, allowing Dirac mass terms.² The free-field neutrino Lagrangian then becomes

$$\mathcal{L}_0 = \sum_i \bar{\nu}_i (i\gamma^\mu \partial_\mu - m_i) \nu_i, \quad (2)$$

where the subscript i labels the neutrino masses. As experience with the weak interactions of the quark sector shows, the fields ν_i appearing in the diagonalized free-field Lagrangian need not be the same as the fields appearing in interactions (e.g., Eq. (1)) in which neutrinos are produced in association with charged leptons. In particular, these fields can be related by

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i, \quad (3)$$

where $U_{\alpha i}$ are the elements of a unitary matrix. In the quark sector, the analogous matrix is known as the CKM (Cabbibo-Kobayashi-Maskawa) matrix [5]. In the neutrino case, it is becoming fashionable to call U the MNS (Maki-Nakagawa-Sakata) matrix [6].

To discern the consequences of a nondiagonal MNS matrix U , consider the following quantum mechanical model of neutrinos propagating in vacuum. (This simplified model is *not* a full quantum field theory of the neutrino. In particular, the neutrino states defined below are not obtained by acting on the vacuum with a creation operator obtained from a quantized neutrino field.) Postulate the existence of a Hilbert space, inhabited by the neutrino state, whose dimension is equal to the number of neutrino flavors. The Hamiltonian is simply the free particle energy. Two relevant bases that span this Hilbert space are (1) a flavor basis $|\nu_\alpha\rangle$, in which the neutrino is produced and detected; and (2) a mass basis $|\nu_i\rangle$, which are eigenstates of the Hamiltonian (with eigenvalues $\sqrt{p^2 + m_i^2}$, taking all neutrino states to have the same three-momentum of magnitude p). Motivated by Eq. (3), take these two bases to be related by

$$\langle \nu_\alpha | \nu_i \rangle = U_{\alpha i}. \quad (4)$$

The neutrino state $|\Psi(t)\rangle$ at time t is

$$|\Psi(t)\rangle = \sum_i e^{-i\sqrt{p^2 + m_i^2}t} |\nu_i\rangle \langle \nu_i | \Psi(0) \rangle. \quad (5)$$

Suppose the neutrino was produced at $t = 0$ in conjunction with a charged lepton of flavor α ; then $|\Psi(0)\rangle = |\nu_\alpha\rangle$. We ask for the probability at time t for the neutrino state to be $|\nu_\beta\rangle$. In the relativistic limit, and using Eq. (4), we find

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = \left| \sum_i U_{\beta i} U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2p}t\right) \right|^2. \quad (6)$$

²Alternatively, an extension to the Higgs sector would allow Majorana mass terms.

With nonzero neutrino masses, a neutrino beam produced in conjunction with charged leptons of flavor α can, when absorbed in a detector, produce charged leptons of flavor β . This method of detecting neutrino mass is typically much more sensitive than direct kinematic searches, because large times between production and detection can overcome small values of $m_i^2/2p$ to produce significant phases in Eq. (6).

Neutrino mixing is altered when the neutrinos pass through matter [7]. Coherent forward scattering with amplitude f off of scatterers with number density N induces an index of refraction n , given by the classic formula

$$n - 1 = \frac{2\pi N}{p^2} f. \quad (7)$$

(The matter background is assumed to be of sufficiently low density that neutrino absorption and non-forward scattering can be neglected.) An index of refraction reduces the neutrino velocity below the speed of light; hence it can also be thought of as an effective squared mass m_{eff}^2 :

$$n - 1 = \frac{m_{\text{eff}}^2}{2p^2}. \quad (8)$$

The effective mass determined by Eqs. (7) and (8) is added to the Hamiltonian of the quantum mechanical model of neutrino evolution. Because the interactions used to compute the forward scattering amplitude f are flavor-dependent, this piece of the Hamiltonian is not diagonal in the mass basis. Diagonalization of this new Hamiltonian yields new effective (“in-medium”) mass eigenstates and a new mixing matrix.

This alteration of flavor mixing in the presence of a matter background can give rise to interesting effects. Suppression or enhancement of flavor mixing can occur, depending on the density of the background, the neutrino momentum, the original (“vacuum”) mixing parameters, etc. In particular, the right combination of background density and neutrino energy induces a resonance, in which even neutrino flavors that hardly mix in vacuum (small off-diagonal terms in $U_{\alpha i}$) become maximally mixed [8]. Eventually, it was recognized [9] that the ν_e produced in the core of the sun by nuclear burning could encounter such a resonance while traversing the solar density gradient. The enhanced flavor mixing in the resonant region would cause a reduced ν_e flux emerging from the sun, complemented by a flux of neutrinos of a different flavor. This is the Mikheyev-Smirnov-Wolfenstein (MSW) effect.

III. CRACKS IN THE FOUNDATION

Clearly, this quantum mechanical model of flavor mixing has given a great deal of insight into interesting physical effects. It has a long and distinguished history, tracing back to the seminal papers on $K^0 - \bar{K}^0$ physics [10], from which the notion of neutrino mixing (or “oscillations”) was suggested by analogy [11]. The model also has some flaws, however.

Consider how the mixing probability of Eq. (6) would be used in describing a neutrino mixing experiment. In a canonical, idealized experiment, neutrinos are produced at a source in association with a charged lepton of flavor α , and measured (with perfect efficiency) with

a detector at a distance L from the source via production of a charged lepton of flavor β . The event rate seen in this experimental setup is

$$d\Gamma_{\alpha\beta} = \int dE_{\mathbf{q}} \left(\frac{d\Gamma_{\alpha,\nu_{\alpha}}}{L^2 d\Omega_{\mathbf{q}} dE_{\mathbf{q}}} \right) (P_{\nu_{\alpha} \rightarrow \nu_{\beta}}) (d\sigma_{\nu_{\beta},\beta}), \quad (9)$$

where the direction of the neutrino momentum \mathbf{q} points from the source to the detector. The first factor in the integrand represents the flux of neutrinos of energy $E_{\mathbf{q}}$ from a process involving a charged lepton of flavor α , and the third factor is the cross section for neutrino detection via a process involving a charged lepton of flavor β . These factors are computed by the standard techniques of quantum field theory (QFT). In contrast, the middle factor is computed with the simplified quantum mechanical model described in the previous section.

Pondering the use of the simplified model's mixing probability in an experimental event rate like Eq. (9) raises some conceptual difficulties:

1. The Hamiltonian evolution describes the time development of the neutrino state; but usually experiments are stationary in time, measuring the mixing probability in space. (Recall that L is the source-detector distance in Eq. (9); the mixing probability as a function of L , not t , is required.)
2. The dynamics of the spin degree of freedom are not addressed. The neutrino states employed in the simplified model of Sec. II do not carry a spin quantum number. If a spin quantum number were included, a formula like Eq. (9) would not properly describe interference between spin states. This is because the parity-violating weak interactions yield different amplitudes for production and detection of the different neutrino spin states. To maintain the factorization of Eq. (9), at best it could only be written as an incoherent sum over the contributions of the different spin states.
3. From a field-theoretic perspective, the whole notion of “flavor eigenstates” is problematic. The usual procedures for computing neutrino production rates and cross sections (the first and third factors in Eq. (9)) involve external particles described as plane waves, resulting in conservation of both energy and momentum. These procedures force the neutrino to have a definite mass, a fact of importance to the interpretation of kinematic searches for neutrino mass [12]. Instead of using interactions with a charged lepton of flavor α in the form of Eq. (1), I propose—and this will be an important point in the discussion of neutrino transport in Sec. V—that such terms be thought of as three separate interactions:

$$\mathcal{L}_I = - \sum_i g_{\alpha i} \bar{\alpha} \gamma^{\mu} (1 - \gamma_5) \nu_i \bar{p} \gamma_{\mu} (1 - g_A \gamma_5) n + \text{H.c.}, \quad (10)$$

where

$$g_{\alpha i} \equiv g U_{\alpha i} \quad (11)$$

is the coupling constant governing the production rate and cross section of a neutrino

of mass m_i in association with a charged lepton of flavor α .³ However, forcing the neutrinos “on-shell” in this way precludes the superposition of mass eigenstates required for coherent flavor mixing. More fundamentally, the fact that flavor states do not have a definite energy (not having a definite mass) creates an algebraic roadblock to attempts to define creation and annihilation operators for flavor states in the standard canonical quantization of field theories⁴ [14]. If creation and annihilation operators cannot be defined for flavor states, the invention of a “flavor basis” for use in the simplified model of Sec. II is physically suspect.

If these difficulties represent cracks in the foundation of neutrino mixing theory, then the relativistic limit is the cement that fills the cracks as a temporary fix:

1. In the relativistic limit, simply take $t \approx L$.
2. The $V - A$ structure of the neutrino interactions ensures that in the relativistic limit there is only one relevant spin degree of freedom: States having the “wrong” helicity are projected out by the factor $(1 - \gamma_5)$ in, for example, interactions like Eq. (1).
3. The production and detection regions are presumably small enough in size in comparison with the source-detector distance that, for $m_i \ll p$, the production and detection regions contribute negligible phase to Eq. (6). The relativistic limit also ensures that the effects of the small masses can be neglected in phase space factors in the interaction rates. Taken together, these considerations allow neutrino mass to be ignored in the production and detection interactions. Moreover, spatially limited production and detection regions imply a spread of momentum, allowing sufficient relaxation of energy-momentum conservation for different mass states to interfere [15]. Finally, the algebraic roadblocks preventing the definition of creation and annihilation operators for flavor states disappear in the ultra-relativistic limit.

These explanations may be sufficient to convince one of the basic physical correctness and quantitative accuracy of the simplified quantum mechanical model in the regime of current experimental relevance (the relativistic limit). Normally, that is enough for physicists—especially experimental physicists! Lingering doubts encroach upon the thoughtful mind, however. At first glance, justification of the replacement $t \approx L$ might be thought of as making a Lorentz transformation from the rest frame of the neutrino to the lab frame: The

³This formulation suggests that the same mechanism that gives neutrinos mass is also responsible for the breakdown of lepton universality.

⁴Creation and annihilation operators for flavor states have been obtained in a nonstandard approach to the canonical quantization of mixing fermion fields [13]. The approach yields Fock spaces for massive and flavor neutrinos that are “unitarily inequivalent representations of the canonical anticommutation relations,” and a flavor vacuum with a nontrivial condensate structure. Because the scattering approach to mixing phenomena reviewed in Sec. IV has a more direct connection to experiment and is theoretically sound, I do not find the case for the physical relevance of the speculative ideas in Refs. [13] to be persuasive.

time t in Eq. (6) might be thought of as that measured by an observer riding with the neutrino between its creation and detection. But in the relativistic limit—so crucial to the above arguments—there is no rest frame of the neutrino. If I then want to consider the small neutrino mass in making this transformation, which of the various mass eigenstates’ rest frames am I transforming from? Presumably they are different: I have chosen equal momenta, but these momenta correspond to different velocities for different masses. More importantly, the production flux and detection cross section in Eq. (9) are not computed in any neutrino rest frame, but in the lab frame; this makes it obvious that this “Lorentz transformation” explanation of $t \approx L$ is inexcusable.

The purist, wanting to believe she truly understands the physics, has a more fundamental question: How would I proceed if the relativistic limit were not applicable? For peace of mind, sometimes it is better to tear down the foundation and rebuild from the ground up.

IV. SCATTERING THEORY APPROACH TO FLAVOR MIXING

A key insight to a more fundamental understanding of flavor mixing is the recognition that, operationally, one does not actually prepare or measure neutrino states. Instead, it is the hadrons and charged leptons that are directly prepared and detected. The neutrinos are merely *intermediate states* in an overall coherent process of neutrino production, propagation, and detection [16].

In this point of view, an experimental (or observational) setup for detecting flavor mixing bears some resemblance to familiar scattering processes treated with Feynman diagrams. At low energies (much smaller than the W^\pm and Z^0 boson masses) where effective point interactions like Eq. (1) are applicable, one can think of the entire neutrino production/propagation/detection process as a tree level Feynman diagram, with a neutrino propagator connecting the external hadrons and charged leptons involved in the production and detection processes. One need not worry about the existence of “flavor eigenstates”, those ill-defined superpositions of mass eigenstates. Instead, the hadronic/charged lepton initial and final states can be connected by each of the (well-defined) neutrino mass eigenstates. Hence the flavor mixing phenomenon is seen to result from the summation of entire diagrams having the same initial and final states, but neutrino propagators of different (definite) mass. This summation of diagrams obviates the need to introduce flavor eigenstates, which, as mentioned earlier, are problematic from a field-theoretic perspective.

Thinking about neutrino flavor mixing in terms of Feynman diagrams might seem strange because of the macroscopic length and time scales involved. When is the last time you thought of a Feynman diagram stretching from a beam dump to a detector tens or hundreds of kilometers away? Or worse yet, from the atmosphere on one side of Earth to a detector on the other side? Or worse still, from the sun to a detector on Earth? In such circumstances it is more natural to imagine the production of a “real” (as opposed to “virtual”) neutrino wave packet, which subsequently travels a macroscopic distance before being absorbed in a detector. And yet the sinusoidal interference terms in Eq. (6)—present in explanations of the atmospheric neutrino problem, and in the “just-so” vacuum oscillation solution to the solar neutrino problem—require just this kind of coherent superposition over macroscopic (even astronomical) distances.

Not surprisingly, the macroscopic distances in the problem produce complications in comparison with the usual use of Feynman diagrams. In the familiar procedure, the external particles in the diagrams are taken to be plane waves. The plane wave scattering amplitude (the S -matrix) contains an overall energy-momentum conserving δ function. In squaring the S -matrix to get a probability, one is therefore faced with a squared δ function. This can be dealt with by localizing the system in a spacetime “box,” resulting in volume and time factors that can be interpreted in such a way as to yield event rates, cross sections, etc. This procedure is really “more a mnemonic than a derivation” [17], however. It is useful because the interactions of interest usually occur (in particle colliders, for instance) in a single, small spacetime volume. It is more convincingly justified, however, by a wave packet description (e.g., Ref. [18]).

In describing neutrino flavor mixing as an overall production/propagation/detection process, it is not possible to compute event rates directly from the plane wave S -matrix with the usual mnemonic described above. This is because a neutrino oscillation experiment involves neutrino production and detection regions which are widely separated in space. In contrast to the case of accelerator particle collisions, the interactions of interest do not all occur in a single volume element. In addition, as also noted above, in this microscopic picture the production and detection of a single neutrino will be separated in time as well as space. In order to describe the spacetime localization one must fall back on a wave packet description of the external particles, in which the amplitude is a superposition of plane wave amplitudes.

Justification of the use of the simplified quantum mechanical model of Sec. II to compute an oscillation probability in general cases involves the following procedure:

1. Write down a properly normalized superposition of plane wave S -matrix amplitudes corresponding to wave packets of hadrons and charged leptons overlapping in regions of limited extent in space and time in the source and detector.
2. Use this amplitude to determine a normalized event rate for the observation of the appropriate external particles in the source and detector.
3. Compare this event rate with Eq. (9), the kind of expression usually used to predict event rates in neutrino mixing experiments. If one can identify factors corresponding to the production flux and detection cross section (the first and third factors in Eq. (9), then everything else is the oscillation probability.

While a number of authors [19] followed the seminal work of Refs. [16] in studying aspects of flavor mixing, this full three-step procedure was set forth, carried to completion, and applied to neutrinos propagating through matter as well as vacuum in Refs. [20,21]. The procedure shows how the mixing probability obtained from the simplified model arises in a more rigorous and physically complete way, and also shows how the interference terms are gradually lost. As a bonus, one sees how the phenomenon would work in nonrelativistic cases, and spin is automatically taken into account through use of the neutrino propagator. In the relativistic limit, factorization into the form of Eq. (9) occurs.

After stripping away the flux and cross section, what remains is the oscillation probability:

$$\begin{aligned}
P_{\nu_\alpha \rightarrow \nu_\beta}(E_{\mathbf{q}}, L) &= \sum_{k, k'} U_{\alpha k} U_{\beta k}^* U_{\alpha k'}^* U_{\beta k'} \\
&\times \exp \left[-i \frac{(m_k^2 - m_{k'}^2) L}{2E_{\mathbf{q}}} - \frac{(m_k^2 - m_{k'}^2)^2 L^2}{32E_{\mathbf{q}}^4 \ell^2} \right].
\end{aligned} \tag{12}$$

Here L is the source-detector distance, which arises naturally from the coordinate space neutrino propagator; unlike the simplified model, no transformation from time to space is necessary. The quantity ℓ is a length scale describing the extent in space and time of the regions of wave packet overlap in the source and detector. The exponential damping factor does not appear in the simplified model of Sec. II (cf. Eq. (6)); it arises because the wave packet treatment of external particles gives rise to intermediate neutrino wave packets which, having different velocities, begin to separate as the source-detector distance is traversed. This damping of interference terms probably only comes into play when L is much larger than the period of flavor oscillations, a case in which binning over energy and source/detector positions tends to wash out the interference terms anyway. Still, it is nice to see that the intrinsic decoherence arises naturally. Finally, the expression from the full calculation of Ref. [20] corresponding to Eq. (9) automatically contains the time delay between neutrino emission and detection, an effect that would have to be added by hand when using the simplified model.

A similar calculation has been performed in the case of neutrinos propagating through a matter background [21]. A goal of this work was to justify the use of the Schrödinger equation of the simplified model, in which the (varying) effective mass induced by neutrino forward scattering is included in the Hamiltonian. Unlike the vacuum case—where the explicit vacuum neutrino propagator was exploited from the outset—an explicit propagator was not initially assumed in the calculation of Ref. [21]. Instead, it was first determined what the general form of the coordinate space neutrino propagator must be in order for the factorization of Eq. (9) to occur. Guided by this expected form of the propagator, the case of a spatially varying background was studied in the relativistic limit, and it was shown that the factor from the neutrino propagator giving rise to the mixing amplitude does in fact obey (a spatial version of) the Schrödinger equation employed in the simplified model of Sec. II.

V. A LESSON TO TAKE AWAY

While the simplified quantum mechanical model of neutrino flavor mixing phenomena reviewed in Sec. II captures the heart of the effect under special (though experimentally relevant) conditions, the scattering-process approach to neutrino flavor mixing phenomena is physically realistic. The fact that it shows how to proceed even in nonrelativistic situations—or with more general interactions than the usual $V - A$ case—shows that a deeper understanding has been attained. One now appreciates that a separate, simplified model of neutrino flavor mixing is a special situation; consideration of the more general picture provides the realization that flavor mixing is a coherent process in which the mechanisms of production and detection of the “particle mixture” are entangled. That Nature should make the oscillation length of two of her neutrino flavors coincide with the diameter

of Earth is already startling; to realize now that Super-Kamiokande is observing a coherent superposition of Feynman diagrams the size of Earth’s diameter is truly amazing!

Beyond the satisfaction of understanding, however, lies a lesson of practical use in the way neutrino transport might be treated in astrophysical phenomena such as supernovae.

The simplified quantum mechanical model described in Sec. II assumes free streaming (or free streaming + coherent forward scattering) neutrinos, but in astrophysical situations interaction rates may exceed or compete with the flavor oscillation period. Neutrino scattering, production, and absorption events are incoherent processes that interrupt coherent flavor oscillations and drive the system to chemical (flavor) equilibrium. A density matrix, with its ability to describe partial coherence, is a natural construct to employ in describing the combined effects of flavor oscillations and interactions. Such an approach has been developed over the years ([22,23]; for a review see Ref. [24]), in which the quantum state vectors used to construct a density matrix are the flavor/mass eigenstates of the simplified quantum model of Sec. II. This approach works in the relativistic limit, and has been used to study effects of flavor mixing during Big Bang nucleosynthesis, in the cores of supernovae, and in the tenuous wind environment outside a nascent neutron star. Each of these environments is either homogeneous or quasi-stationary, and this suits them to a treatment (the density matrix approach of Refs. [22–24]) deriving from the simplified model of Sec. II, which allows evolution in time *or* space to be followed. However, it is not clear how or if this approach could be adapted for use in, say, a dynamic supernova simulation, in which variation in time *and* space must be followed.

In the brave new world of empirically confirmed neutrino mass, the question of how to cope with neutrino transport with flavor mixing acquires a new urgency. Is there a way to handle neutrino transport—with *variation in space and time*—with classical methods (i.e. the Boltzmann equation), while still capturing “flavor mixing” physics? In responding to this question, we should remember an important lesson from the more in-depth study of flavor mixing physics outlined above: *Fundamentally, there is no such thing as flavor eigenstates.* I will argue below that under certain conditions that make a classical treatment feasible, we should consider banishing talk of “electron neutrinos,” “muon neutrinos,” and “tau neutrinos;” instead, we should perhaps speak only of “ m_1 neutrinos,” “ m_2 neutrinos,” “ m_3 neutrinos,” etc.⁵ While the seminal papers on $K^0 - \bar{K}^0$ physics [10] introduced the use of a simplified quantum mechanical model for “particle mixtures,” Gell-Mann and Pais were also sensitive to the problem of what the “true particles” are. They noted that an attempt to introduce “quanta” for objects not obeying fundamental conservation laws “can only be a mathematical device that distracts our attention from the truly physical particles.” They pointed out that the word “particle” should be reserved for the objects having a definite lifetime and mass. In the case of neutrinos, “flavor states” are not Lorentz invariant, not having a definite mass. Following the advice of Gell-Mann and Pais, “flavor states” should not be thought of as true particles; in fact, attempts to define “quanta” (creation and

⁵These neutrinos of definite mass should be given more interesting names. Perhaps they could be called ν_m , ν_n , and ν_s , where the letters come from the authors [6] for whom the neutrino mixing matrix is named; or the Brahma, Vishnu, and Shiva neutrinos; or something else sufficiently different from ν_e , ν_μ , and ν_τ .

annihilation operators) for them run into trouble [14]. Classical transport can only deal with “true particles,” not inherently quantum objects like “particle mixtures.” Therefore, I now investigate the possibility that classical transport of neutrinos with mass should, under certain conditions, be done in terms of the “mass eigenstates.”

Let us examine why thinking only in terms of the mass eigenstates—the valid “quanta” or “truly physical particles”—might allow for a classical treatment of transport while still capturing some “flavor mixing” physics. Consider the behavior of the oscillation probability, Eq. (6) or Eq. (12), as the neutrino flight time t or distance L becomes large in comparison with the “oscillation lengths” associated with the various neutrino squared mass differences, $l_{ij}^{\text{osc}} \equiv (2\pi E_{\mathbf{q}})/(m_i^2 - m_j^2)$. In this limit the interference terms are washed away (whether due to intrinsic decoherence—the exponential damping in Eq. (12)—or integration of many oscillations over a finite energy range), so that

$$P_{\nu_\alpha \rightarrow \nu_\beta} \rightarrow \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2. \quad (13)$$

Inserting this mixing probability into the experimental event rate of Eq. (9), we see that this event rate can now be expressed

$$d\Gamma_{\alpha\beta} = \sum_i \int dE_{\mathbf{q}} \left(\frac{d\Gamma_{\alpha, \nu_i}}{L^2 d\Omega_{\mathbf{q}} dE_{\mathbf{q}}} \right) (d\sigma_{\nu_i, \beta}). \quad (14)$$

In arriving at Eq. (14), the factors $|U_{\alpha i}|^2$ and $|U_{\beta i}|^2$ have respectively been absorbed into the production flux and detection cross section, and the sum over intermediate neutrino mass states taken outside their product. The experimental rate to detect different flavors of charged leptons at the source and detector is no longer a single “flavored neutrino” flux, multiplied by an oscillation probability, multiplied by a “flavored neutrino” cross section, as in Eq. (9); it is now a sum of several contributions, each of which is a “massive neutrino” flux times a “massive neutrino” cross section. The “flavored neutrino” fluxes and cross sections in Eq. (9) are computed (using standard plane wave methods) with interactions like Eq. (1); in contrast, the massive neutrino fluxes and cross sections appearing in Eq. (14) are computed with interactions like Eq. (10), in which the mixing parameters $U_{\alpha i}$ have been absorbed into the coupling constants (see Eq. (11)).

As expected, with no interference terms, the situation can be expressed classically—as an incoherent sum. Notice, however, that the “flavor mixing” is still there: Different flavors of charged leptons (α and β) are associated with production and detection. In this incoherent limit, the discussion surrounding Eq. (10) comes into play. The neutrinos are indeed forced “on shell” at production and detection, but each of these massive neutrino types has a coupling (given by Eq. (11)) to each type of charged lepton, so that (an incoherent version of) flavor mixing still operates. Kinematically—that is, for the purpose of simplifying phase space factors in the rates—the relativistic limit can still be taken. The production rates and cross sections in Eq. (14) will then be the same as the massless case, except for a difference in coupling constants. *But even if the mass can be neglected for phase space purposes, remembering that the “true particles” are the mass states makes all the difference in arriving at an experimental rate that retains flavor mixing effects in the classical limit.*

The validity of using the classical “flavor mixing” rate, Eq. (14), may be challenged: It appears to conflict with the landmark experiment at Brookhaven in 1962 [25] interpreted

as establishing the existence of the ν_μ as a separate neutrino species. Neutrinos from pion decay (with an associated antimuon) were observed to produce muons in a detector, and not electrons. In principle there are two possible reasons for this. First, it may be that Eqs. (13) and (14) are applicable, but that the MNS matrix $U_{\alpha i}$ is very close to diagonal. In this case, one neutrino mass state would have a strong coupling to the muon (see Eq. (11)) and be produced in abundance; but it would have a small coupling to the electron, such that the rate of appearance of electrons in the detector would be below experimental sensitivity. While one of the other neutrino mass states would have a strong coupling to the electron, this state would only be weakly coupled to the muon, so that too few of these neutrinos would be produced in the source to cause a noticeable electron appearance rate in the detector.

A second possible interpretation of the landmark Brookhaven experiment is that that Eqs. (13) and (14) are not applicable; this would be the case if the source-detector distance L is much smaller than the “oscillation lengths” l_{ij}^{osc} associated with the various neutrino squared mass differences. In this case, the mixing probability reduces to

$$P_{\nu_\alpha \rightarrow \nu_\beta} \rightarrow \delta_{\alpha\beta}, \quad (15)$$

so that the experimental flavor mixing rate becomes

$$\begin{aligned} d\Gamma_{\alpha\beta} &= \int dE_{\mathbf{q}} \left(\frac{d\Gamma_{\alpha,\nu_\alpha}}{L^2 d\Omega_{\mathbf{q}} dE_{\mathbf{q}}} \right) (d\sigma_{\nu_\alpha,\alpha}), & (\alpha = \beta) \\ &= 0, & (\alpha \neq \beta), \end{aligned} \quad (16)$$

where the “flavored neutrino” production flux and detection cross section are computed with interactions like Eq. (1). Current phenomenology [1] of both atmospheric and solar neutrinos suggests that this second possibility is the correct interpretation. Both favor strong mixing (large off-diagonal terms in $U_{\alpha i}$), which would have caused observable flavor mixing if Eqs. (13) and (14) were applicable. Atmospheric and solar neutrino data also indicate that neutrino squared mass differences are too small to have been probed by the source-detector distance in the Brookhaven experiment.

The application of these considerations to neutrino transport comes from a correspondence between a flavor mixing experiment and neutrino processes in an astrophysical environment: Microscopic neutrino emission, absorption, and scattering events in an astrophysical environment are analogous to the processes occurring in the neutrino production and detection regions of a flavor mixing experiment, while the neutrino mean free path in an astrophysical setting corresponds to the experiment’s source-detector distance.

Based on this correspondence, one concludes that a classical (Boltzmann equation) treatment is possible if the neutrino mean free path is either much longer or much shorter than the oscillation length. The long mean free path case involves complete incoherence, entailing the physics discussed in connection with Eqs. (13) and (14). In particular, with long mean free paths—and in the absence of coherent forward scattering effects—one would think in terms of the neutrino vacuum mass states, and rewrite all interactions in the manner of Eq. (10), with the mixing matrix elements $U_{\alpha i}$ absorbed into the coupling constants, as in Eq. (11). These revised interactions would then enter into the emissivities and opacities appearing in the Boltzmann equation. On the other hand, the short mean free path case involves

full coherence, but without time for phase differences between the contributing intermediate states to be established, as discussed in connection with Eqs. (15) and (16). For short mean free paths, one can pretend that the neutrinos are massless “flavor states,” and employ the standard electroweak interactions to derive emissivities and opacities.

When effective contributions to neutrino effective mass from forward scattering off a background medium are considered, there are two complications to the proposed classical treatment in the fully incoherent (long mean free path) limit. First, the relationship between flavor fields and the “true particle” quanta is not as simple as in the vacuum case, which involves an overall matrix relation (i.e., Eq. (3) between flavor fields and “mass eigenstate in matter” fields. Second, a medium with varying density—in which resonant enhancement of flavor mixing can occur, as discussed in Sec. II—introduces a new length (and/or time) scale into the problem.

When background matter effects are included in the Hamiltonian of the simplified model of Sec. II, the newly diagonalized Hamiltonian yields “mass eigenstates in matter” that differ from the vacuum mass eigenstates. These new eigenstates should now be considered the “true particles,” playing the same role the vacuum mass eigenstates played in the previous discussion of the incoherent (long mean free path) case.⁶ As discussed previously, if the oscillation length is short compared with the mean free path, a classical (Boltzmann) treatment of transport may be justified, where now distribution functions are defined for the “mass eigenstates in matter.” However, a complication arises due to the fact that the effective mass contribution of the matter background depends on the neutrino momentum. This causes the new “in-matter” mixing matrix (the unitary matrix that diagonalizes the Hamiltonian of the simplified model) to depend on the neutrino momentum as well, so that there is no overall matrix relation like Eq. (3) between flavor fields and “mass-eigenstate in-matter” fields. In a deeper picture, quantization of the neutrino field in a background can still be accomplished [27], though the mixing matrix must now reside inside the integral over the momentum variable of the quantized neutrino field.

By rewriting interactions like Eq. (1) in terms of these quantized neutrino-in-matter fields, it should still be possible to redefine the emissivities and opacities to describe the emission and absorption processes in terms of the new “true particles.” In particular—and this is a crucial point, as discussed below—the *coupling constants describing the interaction strength between the “mass-in-medium” neutrino states and the charged leptons will be determined by the “in-medium” mixing matrix, in a manner conceptually similar to Eq. (11).*

Another complication concerns the fact that in the supernova environment the matter background varies in space and time. To deal with this, one can consider a coarse graining into macroscopically small but microscopically large elements, each with a constant mat-

⁶This borrows from the point of view of condensed matter physics, in which describing systems in terms of “quasi-particles” often provides tremendous simplification. In describing the observation of a new class of excitations called “orbitons,” it was noted that “The prefix quasi is unnecessary. These particles are as real as the more familiar photons, electrons and protons” [26].

ter background.⁷ Each coarse-grained element, having a different background density, will have its own values of the in-medium mass eigenvalues. However, if the variation of the background from one element to another (and one time step to the next) is small compared with the local oscillation length (and time), one can (along the lines of Ref. [28]) consider the ordered mass eigenvalues in adjacent elements to correspond to the same “true particle” types.⁸

Hence, in a Boltzmann treatment in the incoherent (long mean free path) limit, one would follow the distribution functions of the highest mass in-medium neutrino, that of next-highest mass, etc. Even though these masses are slowly changing, one can still consider that one is following the “true particles.”

Note that this procedure automatically reproduces the MSW effect. The key to seeing this is to note that, as mentioned above, the coupling strengths of the mass-in-medium neutrinos to the charged leptons are determined by the in-medium mixing matrix. This mixing matrix—which diagonalizes the effective neutrino mass matrix including background matter effects—will vary with changing density, leading to changes in the relative coupling strengths of the in-medium mass eigenstates. Therefore, the MSW “flavor transformation” is manifest in the changes in the relative strengths of coupling of a given in-medium definite mass neutrino type to the various charged leptons at different positions and times. These local coupling constants will be incorporated into the emissivities and opacities of the in-medium neutrino species, so that evolution of the Boltzmann equation automatically tracks to propensity of the distribution functions of these “true particles” to interact with the various flavors of charged leptons.

The “adiabatic” treatment—in which the local oscillation length (and time) is assumed small in comparison with the scale of variations (in space and time) in the background—can break down when two different mass eigenvalues approach each other as the background varies. In particular, there can be a large probability for a neutrino of one mass eigenvalue to “jump” or tunnel to the adjacent mass eigenvalue. This can be treated as a classic level-crossing problem [29], and the resulting “jump probability” might be incorporated into the Boltzmann equation as an ad-hoc interaction transferring one kind of “true particle” (say, in-medium mass eigenstate 1) into another “true particle” type (say, in-medium mass eigenstate 2).

Focus on the “true particles”—the in-medium mass eigenstates—has a subsidiary benefit in simplifying the treatment of the contribution of background neutrino populations to the neutrino effective mass. It has been thought necessary to treat this effect with a density matrix, because this interaction appears to have off-diagonal contributions in the “flavor basis” (e.g. Refs. [23,24]). However, it was shown in Ref. [21] that in the basis of the “true

⁷Such a coarse graining is analogous to the way one develops a concept of local thermodynamic equilibrium. In the context of a computer simulation it would be natural to identify each coarse-grained element with a zone of the discretized system.

⁸The validity of this approach in regions of rapidly varying background in supernovae—such as the “density cliff” at the surface of the nascent proto-neutron star, and the shock created during the “bounce” of the collapsing core—will have to be examined in more detail.

particles” this contribution could be expressed in terms of a macroscopic neutrino number current.⁹ (In hindsight, it makes sense that the contribution from the “true particles” should be expressible in terms of a classical object!) Hence, being analogous to the background contribution of, say, electrons, treatment of the neutrino background does not by itself require a density matrix approach.

In summary, the lesson to take away is this: The realization from a deeper study of flavor mixing physics that there is really no such thing as “flavor eigenstates” may have practical utility in environments like supernovae, where neutrino transport (ranging from diffusion to free-streaming) must be dealt with in space *and* time. We do not usually need interference terms in astrophysical environments. Needing only classical probabilities, we can hope for a classical Boltzmann equation treatment. When the neutrino mean free path is much smaller than the oscillation lengths, the neutrinos can be treated as massless “flavor states,” with emissivities and opacities derived from the standard electroweak interactions. For the case of a mean free path much longer than the oscillation lengths, I have argued that a classical Boltzmann treatment is still possible *and conceptually simple*—while still retaining “flavor mixing” physics—when the distribution functions are taken to describe the “true particles,” the mass eigenstates (vacuum or in-medium as appropriate), and the interactions are rewritten to absorb the mixing matrix elements into local coupling constants.¹⁰

The applicability of these ideas to neutrino transport in supernovae remains to be explored in detail. For “sterile” neutrinos with masses in the keV range (a warm dark matter candidate [31]), the long mean free path limit may apply in all regions and at all times in the supernova environment. For the smaller mass differences of “active” neutrinos indicated by atmospheric and solar neutrino data, the short mean free path limit will obtain in the core, while the long mean free path limit may be realized further out in the envelope. The the viable Boltzmann treatments in these two limits are conceptually different, and a prescription for piecing them together would have to be determined. Ultimately, it may be found that a more fundamental approach to transport with mixing is necessary to deal with such an intermediate region, at least to determine a valid matching prescription.¹¹ In any case, there remains much exciting work in fleshing out this proposed perspective on neutrino transport, and determining its applicability to active-active and active-sterile mixing scenarios that might affect supernova explosions, heavy-element nucleosynthesis, and supernova neutrino

⁹That work assumed—erroneously, I would now argue—that the vacuum mass eigenstates were the relevant “true particles.” But the basic result should stand, that the contribution of the neutrino background is expressible as a macroscopic current when the right basis is used.

¹⁰The reduction to a Boltzmann-type evolution was appreciated in Ref. [30], but in my opinion the continuing emphasis on the “flavor basis” introduces considerable conceptual complication. Moreover, it is not clear that Ref. [30] arrives at something that can deal with transport in space *and* time.

¹¹More fundamental approaches to quantum kinetics might take the form of an extension of the already-developed density matrix approach [22–24], or application of non-equilibrium field theory via Wigner functions [32] or Green functions [33].

signals in terrestrial detectors.

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REFERENCES

- [1] V. Barger, these proceedings; G. Gratta, these proceedings.
- [2] C. Yanagisawa, “Recent Results from Super-Kamiokande,” these proceedings; Y. Fukuda et al. (Super-Kamiokande Collaboration), Phys. Rev. Lett. **81**, 1562 (1998).
- [3] C. Yanagisawa, “Recent Results from K2K,” these proceedings; S. H. Ahn et al. (K2K Collaboration), hep-ex/0103001.
- [4] Q. R. Ahmad et al. (SNO Collaboration), nucl-ex/0106015.
- [5] A parametrization of this matrix for three generations was given by M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973). It generalized the analogous two-generation case explicated by N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963). An important contribution of the Kobayashi and Maskawa work was the recognition that for three generations, the CKM matrix contains a CP-violating phase that cannot be removed by a redefinition of the quark fields.
- [6] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962) suggested a two-generation theory of neutrino mixing. This mixing did not relate the flavor fields to mass fields, however. Instead, the flavor fields were related to neutrino fields which formed bound states with new hypothetical states to create baryons. Nevertheless, the notion of the neutrino as a “particle mixture” was born! In addition to the the phase present in the CKM matrix, the three-generation MNS matrix for Majorana neutrinos contains two additional phases whose values affect processes in which lepton number changes by two units.
- [7] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
- [8] V. Barger, K. Whisnant, S. Pakvasa, and R. J. N. Phillips, Phys. Rev. D **22**, 2718 (1980).
- [9] S. P. Mikheyev and A. Yu. Smirnov, Nuovo Cimento **9C**, 17 (1986).
- [10] M. Gell-Mann and A. Pais, Phys. Rev. **97**, 1387 (1955); A. Pais and O. Piccioni, Phys. Rev. **100**, 1487 (1955).
- [11] B. Pontecorvo, Sov. Phys. JETP **6**, 429 (1958); **7**, 172 (1958); **26**, 984 (1968); V. Gribov and B. Pontecorvo, Phys. Lett. **B28**, 493 (1969).
- [12] R. E. Shrock, Phys. Lett. **96B**, 159 (1980); Phys. Rev. D **24**, 1232 (1981); **24**, 1275 (1981).
- [13] M. Blasone and G. Vitiello, Ann. Phys. (N.Y.) **244**, 283 (1995); **249**, 363(E) (1996); K. Fujii, C. Habe, and T. Yabuki, Phys. Rev. D **59**, 113003 (1999); **60**, 099903(E) (1999); M. Blasone and G. Vitiello, Phys. Rev. D **60**, 111302 (1999); K. Fujii, C. Habe, and T. Yabuki, Phys. Rev. D **64**, 013011 (2001).
- [14] C. Giunti, C. W. Kim, and U. W. Lee, Phys. Rev. D **45**, 2414 (1992).
- [15] B. Kayser, Phys. Rev. D **24**, 110 (1981).
- [16] C. Giunti, C. W. Kim, J. A. Lee, and U. W. Lee, Phys. Rev. D **48**, 4310 (1993); J. Rich, Phys. Rev. D **48**, 4318 (1993).
- [17] S. Weinberg, *The Quantum Theory of Fields: Volume I, Foundations* (New York, Cambridge University Press, 1995), p. 134.
- [18] M. L. Goldberger and K. M. Watson, *Collision Theory* (New York, John Wiley & Sons, 1964).
- [19] W. Grimus and P. Stockinger, Phys. Rev. D **54**, 3414 (1996); J. E. Campagne, Phys. Lett. **400B**, 135 (1997); K. Kiers and N. Weiss, Phys. Rev. D **57**, 4418 (1998); C. Giunti,

- C. W. Kim, and U. W. Lee, Phys. Lett. **421B**, 237 (1998); W. Grimus, P. Stockinger, and S. Mohanty, Phys. Rev. D **59**, 013011 (1999); A. Ioannisian and A. Pilaftsis, Phys. Rev. D **59**, 053003 (1999).
- [20] C. Y. Cardall, Phys. Rev. D **61**, 073006 (2000).
 - [21] C. Y. Cardall and D. H. J. Chung, Phys. Rev. D **60**, 073012 (1999).
 - [22] A. D. Dolgov, Sov. J. Nucl. Phys. **33**, 700 (1981); M. A. Rudzsky, Astrophys. Space Sci. **165**, 65 (1990).
 - [23] G. Sigl and G. Raffelt, Nucl. Phys. B **406**, 423 (1993); Y.-Z. Qian and G. M. Fuller, Phys. Rev. D **51**, 1479 (1995).
 - [24] G. G. Raffelt, *Stars as Laboratories for Fundamental Physics* (University of Chicago Press, Chicago, 1996), Chap. 9.
 - [25] G. Danby et al., Phys. Rev. Lett. **9**, 36 (1962).
 - [26] P. B. Allen and V. Perebeinos, Nature **410**, 155 (2001).
 - [27] P. D. Mannheim, Phys. Rev. D **37**, 1935 (1988).
 - [28] H. A. Bethe, Phys. Rev. Lett. **56**, 1305 (1986).
 - [29] W. C. Haxton, Phys. Rev. Lett. **57**, 1271 (1986).
 - [30] R. R. Volkas and Y. Y. Y. Wong, Phys. Rev. D **62**, 093024 (2000).
 - [31] A. D. Dolgov and S. H. Hansen, hep-ph/0009083; K. Abazajian, G. M. Fuller, and M. Patel, Phys. Rev. D **64**, 023501 (2001).
 - [32] M. Sirera and A. Pérez, Phys. Rev. D **59**, 125011 (1999).
 - [33] S. Yamada, Phys. Rev. D **62**, 093026 (2000).